Borel-Serre Compactification and Reduction to Liftability

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1 Outline

Our goal today is to reduce the main theorem of [4] to a theorem on lifting mod p cohomology classes. These are as follows.

Main Theorem. Let X be a locally symmetric space for $\mathrm{GL}_n/\mathbb{Q}$. Let ψ be a system of Hecke eigenvalues in $H^i(X, \overline{\mathbb{F}}_p)$. Then there is a continuous semisimple Galois representation

$$\rho_\psi : \mathrm{Gal}(\overline{F}/F) \to \mathrm{GL}_n(\overline{\mathbb{F}}_p)$$

whose Frobenius eigenvalues match the Hecke eigenvalues of ψ.

Lifting Theorem. Let X' be a Shimura variety of Hodge type, and let ψ be a system of eigenvalues in $H^i_c(X', \overline{\mathbb{F}}_p)$. Then there exists a cuspidal Hecke eigenform f such that the Hecke eigenvalues of f are congruent to ψ mod p.

The point is that it's known how to associate Galois representations to cuspidal Hecke eigenforms, so to get a Galois representation from a Hecke eigensystem in $H^i(X, \overline{\mathbb{F}}_p)$ we need only produce a corresponding eigensystem in $H^i(X', \overline{\mathbb{F}}_p)$. Then the second theorem allows us to lift to a cuspidal Hecke eigenform, to which we can associate a Galois representation, and then reduce mod p to get our desired Galois representation.

The key tool we'll use to transfer cohomology from X to X' is Borel-Serre compactification. More precisely, we'll take $X' = X_G$ to be a locally symmetric space for $G = \mathrm{Sp}_{2n}$, i.e. a Siegel modular variety. Then $M = \mathrm{GL}_n$ is the Levi subgroup of a maximal parabolic subgroup P in G, and the corresponding locally symmetric spaces fit into the following diagram

$$\begin{tikzcd}
X_P \arrow[r, "\text{open}"] \arrow[d, bend right, "\text{torus bundle}"'] & \partial X_G^{\mathrm{BS}} \arrow[r, "\text{bound'ry}"] & X_G^{\mathrm{BS}} & X_G \arrow[l] \\
X_M \arrow[u, bend right, "\text{section}"'] & & &
\end{tikzcd}$$

We'll use these maps to get maps on cohomology compatible with Hecke actions, which will allow us to transport around Hecke eigensystems.

2 Borel-Serre Compactification

First let's discuss Borel-Serre compactification, which is a procedure that gives a compactification of a locally symmetric space for a reductive group as a manifold with corners. Our discussion follows [3].

Fix $G/\mathbb{Q}$ a connected reductive algebraic group, $K \subset G(\mathbb{R})$ a maximal compact subgroup, and $\Gamma \subset G(\mathbb{Q})$ an arithmetic subgroup. A nice illustrative example to keep in mind is $G = \mathrm{SL}_2$, with $K = \mathrm{SO}_2(\mathbb{R}) \subset \mathrm{SL}_2(\mathbb{R})$ and $\Gamma \subset \mathrm{SL}_2 \mathbb{Q}$ a congruence subgroup.

The center of G is an algebraic torus over $\mathbb{Q}$, which contains a greatest $\mathbb{Q}$-split subtorus A_G. Define

$$^0G = \bigcap_{\chi} \ker(\chi^2)$$

where $\chi : G \to \mathbb{G}_m$ runs over rational characters of G. This is again a connected reductive group over $\mathbb{Q}$, and the group of real points of G decomposes as

$$G(\mathbb{R}) = {}^0G(\mathbb{R}) \times A_G(\mathbb{R})^+$$

(where the $+$ superscript denotes the topological identity component).

Our symmetric space for G is

$$D = G(\mathbb{R})/K \cdot A_G(\mathbb{R})^+$$

and our locally symmetric space is

$$X = \Gamma\backslash D = \Gamma\backslash G(\mathbb{R})/K \cdot A_G(\mathbb{R})^+.$$

For convenience, we can actually always assume A_G is trivial; this doesn't affect the formation of our locally symmetric space because $G(\mathbb{R})/K \cdot A_G(\mathbb{R})^+ = {}^0G(\mathbb{R})/K$ (and we can replace G by ${}^0G$).

In our example of SL_2, the center is $\{\pm 1\}$ so contains no non-trivial torus, and our symmetric spaces are

$$D = \mathrm{SL}_2(\mathbb{R})/\mathrm{SO}_2(\mathbb{R}) = \mathfrak{H} \quad \text{and} \quad X = \Gamma\backslash \mathrm{SL}_2(\mathbb{R})/\mathrm{SO}_2(\mathbb{R}),$$

the classical upper half-plane and modular curves.

The Borel-Serre compactification of X is got by adjoining boundary components to D to get a "partial compactification", and then quotienting by Γ to get a true compactification of X. The boundary components we add to D correspond to parabolic subgroups of G.

Let P be a proper rational parabolic subgroup of G. Then we have a Levi decomposition

$$P = U_P \rtimes L_P$$

where U_P is the unipotent radical of P and L_P is a Levi subgroup. As above we have $L_P(\mathbb{R}) = {}^0L_P(\mathbb{R}) \times A_P(\mathbb{R})^+$ where A_P is the largest $\mathbb{Q}$-split torus in the center of L_P. Thus

$$P(\mathbb{R}) = U_P(\mathbb{R}) \cdot {}^0L_P(\mathbb{R}) \cdot A_P(\mathbb{R})^+.$$

From the Iwasawa decomposition of G one can see that P acts transitively on D, which is to say

$$D = G(\mathbb{R})/K = P(\mathbb{R})/K_P$$

(where $K_P = K \cap P(\mathbb{R})$). Now we can define a right action of $A_P(\mathbb{R})^+$ on D by $(g \cdot K_P)a = ga \cdot K_P$ ($g \in P(\mathbb{R})$ and $a \in A_P(\mathbb{R})^+$), which is well defined because $A_P(\mathbb{R})^+$ commutes with $K_P \subset {}^0L(\mathbb{R})$.

Define the Borel-Serre boundary component associated to P to be $e_P = D/A_P(\mathbb{R})^+$, and the Borel-Serre compactification D^{BS} to be (as a set) the disjoint union of D and e_P for each proper rational parabolic P.

Topologising D^{BS} is a complicated affair, which I won't discuss in detail.

Theorem (Borel-Serre)**.** *There is a topology on D^{BS} (the Satake topology) such that the action of $G(\mathbb{Q})$ on D extends to an action by homeomorphisms on D^{BS}.*

But there is the following nice intuition. The orbits of $A_P(\mathbb{R})^+$ acting on D are totally geodesic submanifolds, and the boundary component $e_P = D/A_P(\mathbb{R})^+$ is attached as the set of limit points of these orbits.

Let's return again to the case of SL_2. It has only one parabolic subgroup up to conjugacy, namely the upper-triangular matrices

$$P = \left\{ \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} \right\} \subset \mathrm{SL}_2$$

with Levi decomposition $P = U_P \rtimes L_P$ where

$$U_P = \left\{ \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \right\} \quad \text{and} \quad L_P = A_P = \left\{ \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \right\}.$$

Note that here the whole Levi is a $\mathbb{Q}$-split torus. Our symmetric space, the upper half-plane, is given by

$$\begin{gathered} P/K_P \xrightarrow{\sim} \mathfrak{H} \\ \begin{pmatrix} y^{1/2} & xy^{-1/2} \\ 0 & y^{-1/2} \end{pmatrix} \mapsto \begin{pmatrix} y^{1/2} & xy^{-1/2} \\ 0 & y^{-1/2} \end{pmatrix} i = x + yi \end{gathered}$$

The action of A_P is given by

$$(x+yi) \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} = \begin{pmatrix} y^{1/2} & xy^{-1/2} \\ 0 & y^{-1/2} \end{pmatrix} \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} i = \begin{pmatrix} ay^{1/2} & a^{-1}xy^{-1/2} \\ 0 & a^{-1}y^{-1/2} \end{pmatrix} i = x + a^2 yi,$$

whose orbits are vetrtical half-lines. Thus D/A_P is a line, which we glue to the top of the upper half-plane. The same procedure is performed at each cusp, so that D^{BS} is the upper half-plane together with countably many lines, one glued at each cusp. Compare this to the usual (Baily-Borel) compactification, which glues a single point at each cusp. (Here and later I'm abusing the word "cusp" a bit, but you know what I mean).

Back to the general case. The Borel-Serre compactification of $X = \Gamma \backslash D$ is then

$$X^{\mathrm{BS}} = \Gamma \backslash D^{\mathrm{BS}}$$

and the quotients Y_P of e_P are its boundary components. The action of Γ can identify boundary components and identify points within boundary components, but is essentially well behaved on the boundary. To be precise, setting $\Gamma_P = \Gamma \cap P \subset \Gamma$, the covering

$$\Gamma_P \backslash D^{\mathrm{BS}} \to \Gamma \backslash D^{\mathrm{BS}}$$

is in fact one-to-one near the boundary component e_P, so that points of e_P are identified by Γ only as much as by Γ_p. The resulting X^{BS} is a compact manifold with corners.

Theorem (Borel-Serre). *X^{BS} is compact, and the inclusion $X \to X^{\mathrm{BS}}$ is a homotopy equivalence. X^{BS} is stratified with finitely many strata Y_P corresponding to the Γ-conjugacy classes of rational parabolic subgroups $P \subset G$. Each Y_P has a neighborhood V diffeomorphic to $Y_P \times (0,\infty]^r$ (where r is the rank of A_P) whose faces $Y_P \times (0,\infty)^s$ are the intersections $Y_Q \cap V$ for some $Q \supset P$.*

Note that the boundary components

$$Y_P = \Gamma_P \backslash P(\mathbb{R}) / K_P \cdot A_P(\mathbb{R})^+$$

are locally symmetric spaces for the parabolic subgroups of G, and that larger parabolics produce higher dimensional boundary components. In particular, the locally symmetric space of a maximal parabolic $P \subset G$ is an open subset of the boundary of the Borel-Serre compactification of X.

Returning to our example: we saw that D^{BS} is the upper half-plane with a line glued to each cusp. The quotient by a congruence subgroup Γ identifies the lines glued at Γ-equivalent cusps, and (as is clear in the case of the line at infinity) rolls each line into a circle. The resulting X^{BS} is a modular curve compactified by adding finitely many boundary circles. Compare this to the usual (Baily-Borel) compactification, where the circle are replaced by points.

3 Reduction to Liftability

Equipped with the Borel-Serre compactification, let's now see how to reduce the main theorem to lifting mod p cohomology. We'll only discuss the case $F = \mathbb{Q}$; some modifications are necessary for F totally real or CM, but the main ideas are identical.

Set $G = \mathrm{Sp}_{2n}/\mathbb{Q}$. This contains $M = \mathrm{GL}_n/\mathbb{Q}$ via

$$\begin{aligned} \mathrm{GL}_n &\to \mathrm{Sp}_{2n} \\ M &\mapsto \begin{pmatrix} M & 0 \\ 0 & {}^tM^{-1} \end{pmatrix} \end{aligned}$$

as the Levi subgroup of a maximal parabolic $P \subset G$. Taking $K_M \subset M(\mathbb{A}_f)$ compact open, $K_\infty \subset M(\mathbb{R})$ maximal compact, and $\mathbb{R}^+ \subset M(\mathbb{R})$ the positive scalar matrices, we have a corresponding locally symmetric space

$$X_M = M(\mathbb{Q}) \backslash [(M(\mathbb{R})/\mathbb{R}^+ K_\infty) \times M(\mathbb{A}_f)/K_M]$$

and similarly X_P for P and X_G for G. This is not the same notion of locally symmetric space as the one used in the previous section; actually the locally symmetric space here is a disjoint union of finitely many locally symmetric spaces in the previous sense, so the departure is not dramatic, and we will not dwell on showing precisely the same facts in this context.

The Main Theorem seeks to associate a Galois representation to a system of Hecke eigenvalues in $H^i(X_M, \overline{\mathbb{F}}_p)$; the Lifting Theorem associates a Galois representation to a system of Hecke eigenvalues in $H^i(X_G, \overline{\mathbb{F}}_p)$. Recall our idea is to transfer Hecke eigenvalues between these different cohomology groups.

To do this we'll need to investigate the relations between X_M, X_P, and X_G, and in particular their cohomology. We'll need some compatibility in how we form these spaces; if $K_G \subset G(\mathbb{A}_f)$ is the compact open defining X_G, then we should take $K_P = K_G \cap P(\mathbb{A}_f)$ to be the compact open for X_P, and take $K_M = K_P \cap M(\mathbb{A}_f)$ to be the compact open for X_M (we also assume that K_M agrees with the image of K_P under the Levi quotient).

Let X_G^{BS} be the Borel-Serre compactification, and $\partial X_G^{\mathrm{BS}} = X_G^{\mathrm{BS}} \backslash X_G$ the boundary. From our above discussion of the construction, we get an open embedding

$$X_P \hookrightarrow \partial X_G^{\mathrm{BS}}.$$

There is also a natural long exact sequence associated to a manifold with corners:

$$\cdots \to H_c^i(X_G, \mathbb{Z}/p^m) \to H^i(X_G^{\mathrm{BS}}, \mathbb{Z}/p^m) \to H^i(\partial X_G^{\mathrm{BS}}, \mathbb{Z}/p^m) \to \cdots,$$

where we can replace X_G^{BS} by X_G in the middle term because the inclusion $X_G \hookrightarrow X_G^{\mathrm{BS}}$ is a homotopy equivalence. Combining all these facts we obtain natural maps

$$H_c^i(X_P, \mathbb{Z}/p^m) \to H^i(\partial X_G, \mathbb{Z}/p^m) \to H^i(X_P, \mathbb{Z}/p^m).$$

Regarding X_M, we have the following maps.

Lemma. *The quotient $P \to M$ induces a natural $(S^1)^k$-bundle*

$$X_P \to X_M,$$

where k is the dimension of the unipotent radical of P. In addition, the inclusion $M \subset P$ induces embedding $X_M \hookrightarrow X_P$ which is a section of the torus bundle.

We can use this to induce maps in both directions between the cohomology of X_M and X_P.

Putting all these facts together results in the following commutative diagram (of $\mathbb{Z}_p$-modules).

$$\begin{tikzcd}
 & H^i(\partial X_G^{\mathrm{BS}}, \mathbb{Z}/p^m) \arrow[dr] & \\
H_c^i(X_P, \mathbb{Z}/p^m) \arrow[ur] \arrow[rr] & & H^i(X_P, \mathbb{Z}/p^m) \arrow[d] \\
H_c^i(X_M, \mathbb{Z}/p^m) \arrow[u] \arrow[rr] & & H^i(X_M, \mathbb{Z}/p^m)
\end{tikzcd}$$

Now let's examine the action of Hecke algebras. A detail we skipped over before is that we should fix a finite set S of places of $\mathbb{Q}$ containing p, and we want our compact opens in G, P, M to have the form $K_G = K_{G,S} K_G^S$ where $K_{G,S} \subset G(\mathbb{A}_{S,f})$ and $K_G^S \subset G(\mathbb{A}_f^S)$ (and similarly for P and M). Define

$$\mathbb{T}_G = \mathbb{Z}_p[K_G^S \backslash G(\mathbb{A}_f^S)/K_G^S]$$

and similarly $\mathbb{T}_P$, $\mathbb{T}_M$.

Recall also that we assumed K_G to be sufficiently small. This implies that all congruence subgroups are torsion free, so that the quotients defining the Borel-Serre compactification are by discontinuous group actions. This shows that the Hecke algebra $\mathbb{T}_G$ acts on the cohomology of the boundary.

$$\mathbb{T}_G \mapsto \mathrm{End}_{\mathbb{Z}/p^m}(H^i(\partial X_G^{\mathrm{BS}}, \mathbb{Z}/p^m))$$

The Hecke algebras $\mathbb{T}_P$ and $\mathbb{T}_M$ also act on the (usual and compactly supported) cohomology of their respective locally symmetric spaces, giving

$$\mathbb{T}_P \to \mathrm{Hom}_{\mathbb{Z}/p^m}(H_c^i(X_P, \mathbb{Z}/m), H^i(X_P, \mathbb{Z}/p^m))$$

and similarly for $\mathbb{T}_M$. Note that these maps do not depend on whether we let the Hecke algebra act on H^i or H^i_c. For M, define also the *interior cohomology*

$$H^i_!(X_M, \mathbb{Z}/p^m) = \mathrm{im}(H^i_c(X_P, \mathbb{Z}/m) \to H^i(X_P, \mathbb{Z}/p^m)).$$

Then $\mathbb{T}_M$ acts also on endomorphisms of the interior cohomology groups.

The maps between cohomology of our spaces are compatible with Hecke actions in the following sense.

Lemma. *The following diagram (of $\mathbb{Z}_p$-modules) is commutative.*

$$\begin{tikzcd}
\mathbb{T}_G \arrow[r] \arrow[d, "\text{restriction}"'] & \mathrm{End}_{\mathbb{Z}/p^m}(H^i(\partial X_G^{\mathrm{BS}}, \mathbb{Z}/p^m)) \arrow[d] \\
\mathbb{T}_P \arrow[r] \arrow[d, "\substack{\text{integration along}\\ \text{unipotent fibers}}"'] & \mathrm{Hom}_{\mathbb{Z}/p^m}(H^i_c(X_P, \mathbb{Z}/m), H^i(X_P, \mathbb{Z}/p^m)) \arrow[d] \\
\mathbb{T}_M \arrow[r] & \mathrm{Hom}_{\mathbb{Z}/p^m}(H^i_c(X_M, \mathbb{Z}/m), H^i(X_M, \mathbb{Z}/p^m))
\end{tikzcd}$$

Corollary. *Let $\overline{\mathbb{T}}_G$ be the image of $\mathbb{T}_G$ in $\mathrm{End}_{\mathbb{Z}/p^m}(H^i(\partial X_G^{\mathrm{BS}}, \mathbb{Z}/p^m))$ and $\overline{\mathbb{T}}_M$ the image of $\mathbb{T}_M$ in $\mathrm{End}_{\mathbb{Z}/p^m}(H^i_!(X_M, \mathbb{Z}/p^m))$. Then the following diagram (of $\mathbb{Z}_p$-algebras) is commutative.*

$$\begin{tikzcd}
\mathbb{T}_G \arrow[r] \arrow[d, "\substack{\text{Satake}\\ \text{transform}}"'] & \overline{\mathbb{T}}_G \arrow[d] \\
\mathbb{T}_M \arrow[r] & \overline{\mathbb{T}}_M
\end{tikzcd}$$

This is essentially the tool that allows us to push around Hecke eigensystems.

To be precise about the Galois representations we get out, we need to make our Hecke algebras and the maps between them more explicit. We have a decomposition into local factors

$$\mathbb{T}_G = \bigotimes_{v \notin S} \mathbb{T}_{G,v}$$

where

$$\mathbb{T}_{G,v} = \mathbb{Z}_p[K_{G,v} \backslash G(\mathbb{Q}_v)/K_{G,v}]$$

and similarly for $\mathbb{T}_P$ and $\mathbb{T}_M$.

Lemma 1. *The Satake transform $\mathbb{T}_G \to \mathbb{T}_M$ is given on each local factor by*

$$\begin{tikzcd}
\mathbb{T}_{G,v}[q_v^{1/2}] \arrow[r, "\text{Satake transform}"] \arrow[d, equal] & \mathbb{T}_{M,v}[q_v^{1/2}] \arrow[d, equal] \\
\mathbb{Z}_p[q_v^{1/2}][X_1^{\pm 1}, \ldots, X_n^{\pm 1}]^{S_n \ltimes (\mathbb{Z}/2)^n} \arrow[r] & \mathbb{Z}_p[q_v^{1/2}][X_1^{\pm 1}, \ldots, X_n^{\pm 1}]^{S_n}
\end{tikzcd}$$

$$X_i \longmapsto q_v^{(n+1)/2} X_i$$

where q_v is the size of the residue field at v.

Rather than dealing directly with representations, the paper uses the language of determinants, which are a pseudo-representation type thing. Precisely, a *determinant* of a group G with values in a ring R is a multiplicative R-polynomial map $D : R[G] \to R$. This induces a map $D : R[X][G] \to R[X]$, and $D(1 - Xg) \in R[X]$ is called the *characteristic polynomial* of g.

If we have a representation $k[G] \to \mathrm{End}(V)$ of G on a k-vector space, then

$$k[G] \to \mathrm{End}(V) \xrightarrow{\det} k$$

is a determinant in the above sense, whose characteristic polynomials are the characteristic polynomials of the representation. The key fact is that if k is an algebraically closed field, then every determinant of G with values in k is associated to a semisimple representation of G on a k-vector space. Furthermore, if we have a determinant with values in R and a map $R \to k$ we can tensor with k to get a determinant with values in k, i.e. a representation.

In the present situation: we'll talk about determinants of $G_{\mathbb{Q},S}$ with values in Hecke algebras. This is essentially the same as attaching representations to Hecke eigensystems, because an eigensystem is a map $\mathbb{T} \to k$, from which we can produce a representation in the above way.

Corollary. *There is a nilpotent ideal $I \subset \overline{\mathbb{T}}_M$ and a continuous determinant D of $G_{\mathbb{Q},S}$ with values in $\mathbb{T}_M/I$ such that*

$$D(1 - X\,\mathrm{Frob}_v) = \tilde{P}_v(X).$$

How to show this: the commuting diagram of Hecke algebras allows us to produce a determinant of $\overline{\mathbb{T}}_M$ (i.e. a rep from $H^i_!(X_M, \mathbb{Z}/p^m)$) from a determinant of $\overline{\mathbb{T}}_G$ (i.e. a rep from $H^i(\partial X_G^{\mathrm{BS}}, \mathbb{Z}/p^m)$). Then from the long exact sequence

$$\cdots \to H^i(X_G, \mathbb{Z}/p^m) \to H^i(\partial X_G^{\mathrm{BS}}, \mathbb{Z}/p^m) \to H_c^{i+1}(X_G, \mathbb{Z}/p^m) \to \cdots$$

it suffices to produce similar determinants with values in

$$\mathrm{im}(\mathbb{T}_G \to \mathrm{End}(H^i(X_G, \mathbb{Z}/p^m))) \quad \text{and} \quad \mathrm{im}(\mathbb{T}_G \to \mathrm{End}(H_c^i(X_G, \mathbb{Z}/p^m))),$$

i.e. representations from $H^i(X_G, \mathbb{Z}/p^m)$. Finally, the explicit map on Hecke algebras tells us the form the characteristic polynomial will take if we start with a Galois rep associated to a Hecke eigensystem in $H^i(X_G, \mathbb{Z}/p^m)$.

This last corollary essentially shows the existence of a Galois representation associated to any Hecke eigensystem in the cohomology of X_M; but it's not quite the right one yet (e.g. in terms of characteristic polynomial). If the representation we're looking for is ρ, then the one we have is something like $\rho \oplus \rho^\vee \oplus 1$. With some more work we can extract the correct Galois representation; i.e., produce a determinant of $G_{F,S}$ with values in $\overline{\mathbb{T}}_M/I$ (for some nilpotent I) such that

$$D(1 - X\,\mathrm{Frob}_v) = P_v(X).$$

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